Lineare Algebra und Analytische Geometrie
Sommersemester 2026
Extra Problems

(1) Let V be a finite dimensional Euklidean space and $f\colon V \to V$ is a linear transformation. Show that $f + f^*$ is diagonalisable.

(2) Let V be a finite dimensional Euklidean space and $f\colon V \to V$ is a linear transformation such that $f^* = f^2$.
   (a) Show that f is normal.
   (b) Show that $f^4 = f$.
   (c) Show that $\det(f) = 1$ or $\det(f) = 0$.

(3) Let V be a finite dimensional Euklidean space and U a subspace of V. Show that the orthogonal projection $\pi\colon V \to V$ onto U is self-adjoint.

(4) Let V be a finite dimensional Euklidean space and $f\colon V \to V$ is a linear transformation such that $f^* = f^2$. Show that if λ is an eigenvalue of f, then $\lambda = 0$ or 1. Conclude that f is an orthogonal projection.

(5) Let V be a finite dimensional Euklidean space and $f\colon V \to V$ is a linear transformation such that $f^* = f^2$. Show that $f = f^2$.

(6) Let V be a finite dimensional Euklidean space and $f\colon V \to V$ is a linear transformation such that $f^* = f^2 + f$.
   (a) Show that f is diagonalisable.
   (b) Show that $f = f^4 + 2f^3 + 2f^2 + f$.

(7) Let $\mathrm{Mat}_{3\times 3}(\mathbb{R})$ denote the space of 3×3 matrices with real entries. Let $A, B, C \in \mathrm{Mat}_{3\times 3}(\mathbb{R})$ such that $\det(A) = 2$, $\det(AB) < 0$ and $\det(BC) > 0$. Show that there does not exist a $D \in \mathrm{Mat}_{3\times 3}(\mathbb{R})$ such that $D^2 = C$.

(8) Let $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ denote the space of 2×2 matrices with real entries. Consider the inner product $\langle A, B\rangle = \mathrm{Spur}(A^T B)$ on $\mathrm{Mat}_{2\times 2}(\mathbb{R})$. Show that if $\mathrm{Spur}(A) = 0$, then A is orthogonal to A^2. Hint: Write

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and compute A^2 and $\langle A, A^2\rangle$. Things will cancel out.

(9) Let $P_2(\mathbb{R})$ denote the space of polynomials with real coefficients up to degree 2. Let $\mathcal{B} = \{1 + x, 1 - x, x^2\}$ be a basis of $P_2(\mathbb{R})$. Find the matrix representation $[f]_{\mathcal{B},\mathcal{B}}$ where $f\colon P_2(\mathbb{R}) \to P_2(\mathbb{R})$ is the differentiation map $f(p) = p'$.

(10) Let $P_2(\mathbb{R})$ denote the space of polynomials with real coefficients up to degree 2 and $f\colon P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the differentiation map $f(p) = p'$. Find f^*.

(11) Let $P_2(\mathbb{R})$ denote the space of polynomials with real coefficients up to degree 2 and $f\colon P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the differentiation map $f(p) = p'$. Is f normal/self-adjoint/unitary?

(12) Let V be a finite dimensional Euklidean space and U a subspace of V. Show that the orthogonal projection $\pi\colon V \to V$ onto U is self-adjoint.

(13) Let $\mathrm{Mat}_{3\times 3}(\mathbb{C})$ denote the space of 3×3 matrices with complex entries. Show that if $A \in \mathrm{Mat}_{3\times 3}(\mathbb{C})$ is diagonalisable, then there exists a $B \in \mathrm{Mat}_{3\times 3}(\mathbb{C})$ such that $B^2 = A$.

More coming.