Linear Algebra and Analytic Geometry
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Chapter 1

Review

In this chapter, we will recall some important concepts from the first semester which we will need in this course. To be more precise, we will recall matrix representations of linear transformations between finite dimensional vector spaces.

1.1 Matrix representations

Given a field k. A linear transformation $f\colon V \to W$ between two finite dimensional k-vectorspaces is a function with the property

$$f(x+y) = f(x+y)$$
$$f(cx) = cf(x)$$

for any $x, y \in V$ and $c \in k$. Given a basis $\alpha = \{v_1, \ldots, v_n\}$ of V and $\beta = \{w_1, \ldots, w_m\}$ of W, we can find a (unique) matrix which *represents* this linear transformation. But what does this mean? What does it mean to *represent* a linear transformation?

1.1.1 Coordinate vectors

We start with an easier question. Let k be a field and V be a finite dimensional k-vectorspace with an *ordered* basis $\mathsf{B} = \{v_1, \ldots, v_n\}$. This means that

1. B *spans* V. That is, any vector $v \in V$ can be written as a linear combination of vectors in V. This means that for any $v \in V$, we can find scalars $a_1, \ldots, a_n$ such that

$$v = a_1 v_1 + \ldots + a_n v_n.$$

2. B *is linearly independent*. This means that if v can be written as a linear combination of vectors in B, then this representation is unique. That is, if we can write the same vector as

$$v = a_1 v_1 + \ldots + a_n v_n$$
$$v = b_1 v_1 + \ldots + b_n v_n$$

then we must have $a_1 = b_1, \ldots, a_n = b_n$.

Therefore, if we have a basis $\mathsf{B} = \{v_1, \ldots, v_n\}$ of the k-vectorspace V and if we can write $v \in V$ as $v = a_1v_1 + \ldots + a_nv_n$, then any information regarding this vector v is given by the scalars $a_1, \ldots, a_n$. We write this as follows.

Definition 1.1.1. We put

$$[v]_{\mathsf{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

where $\mathsf{B} = \{v_1, \ldots, v_n\}$ is a basis of the k-vectorspace V and if we can write $v \in V$ as $v = a_1v_1 + \ldots + a_nv_n$. Note that the column vector $[v]_{\mathsf{B}}$ is an element of k^n. We call it the *coordinate vector of v with respect to* B.

Remark 1.1.2. Note that the coordinate vector of a vector depends on the basis. If one changes the basis, the coordinate vector may or may not change.

Example 1.1.3. Let $V = P_2(\mathbb{R})$ be the space of polynomials with coefficients in $\mathbb{R}$ with degree up to 2. Consider the ordered bases $\mathsf{B}_1 = \{1, x, x^2\}$ and $\mathsf{B}_2 = \{1+x, 1-x, x^2\}$ and the vector $p = 5 + x + 7x^2$. The question

$$\textit{Determine } [p]_{\mathsf{B}_1}.$$

is the same as the question

$$\textit{Find } a_1, a_2, a_3 \in \mathbb{R} \textit{ such that } p = a_1 \cdot 1 + a_2 \cdot x + a_3 \cdot x^2.$$

Therefore, we have

$$[5 + x + 7x^2]_{\mathsf{B}_1} = \begin{bmatrix} 5 \\ 1 \\ 7 \end{bmatrix}.$$

Similarly, to determine $[p]_{\mathsf{B}_2}$, we should find $a, b, c \in \mathbb{R}$ such that

$$5 + x + 7x^2 = a(1+x) + b(1-x) + cx^2.$$

We can write the right hand side as $a+b+(a-b)x+cx^2$ and this gives us a system of linear equations

$$\begin{aligned} a + b &= 5 \\ a - b &= 1 \\ c &= 7 \end{aligned}$$

and by solving this system of linear equations, we find

$$[5 + x + 7x^2]_{\mathsf{B}_2} = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}.$$

So, the same vector p has two different vector representations here.

Lemma 1.1.4. *The function $V \to k^n$ given by the rule $v \mapsto [v]_{\mathsf{B}}$ is a linear transformation. In fact, it is an isomorphism.*

Exercise 1.1.5. Prove this lemma.

Exercise 1.1.6. In class, we discussed that most of linear algebra is translating the language into correct form. Most of the time, by asking the question "What does it mean?" we can reduce our questions to solving a system of linear equations. Discuss the relationship between "B is a basis" and "$v \mapsto [v]_{\mathsf{B}}$ is an isomorphism" from this point of view.

1.1.2 Matrix of a linear transformation

Now, assume that we have a linear transformation $f\colon V \to W$ between two finite dimensional vectorspaces over a field k. Let $\mathsf{B} = \{v_1, \ldots, v_n\}$ be a basis for V and $\mathsf{C} = \{w_1, \ldots, w_m\}$ a basis for W. Then, by Lemma 1.1.4, we have a diagram

$$\begin{array}{ccc} V & \xrightarrow{\quad f \quad} & W \\ \downarrow{\scriptstyle \sigma_{\mathsf{B}}} & & \downarrow{\scriptstyle \sigma_{\mathsf{C}}} \\ k^n & & k^m \end{array}$$

where σ_{B} and σ_{C} are isomorphisms. Since σ_{B} is an isomorphism, it has an inverse σ_{B}^{-1}. Hence, we get the diagram

$$\begin{array}{ccc} V & \xrightarrow{\quad f \quad} & W \\ {\scriptstyle \sigma_{\mathsf{B}}^{-1}}\uparrow & & \downarrow{\scriptstyle \sigma_{\mathsf{C}}} \\ k^n & \dashrightarrow & k^m \end{array}$$

where the dotted map is a linear map $k^n \to k^m$ given by the composition $\sigma_{\mathsf{C}} \circ f \circ \sigma_{\mathsf{B}}^{-1}$. This is again a linear map and we know from the first course in linear algebra that any such map is given by an $m \times n$ matrix with coefficients in k.

Definition 1.1.7. We call the matrix in the previous paragraph *the matrix representation of f with respect to B and C.*

This is a good definition but how do we actually write this matrix? Here, we will talk about the recipe, give an example and then discuss where this recipe comes from.